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On Linearoid Differential Equations.

By Dr. E. J. Wilczynski.

The fundamental notions of the theory of linear differential equations can be applied to a very large category of non-linear differential equations. In a former paper* I have shown, in general, how this may be done, the existence of the differential equations, however, not being demonstrated. Moreover, the point of view in that paper was somewhat different, and the results obtained there are put into a clearer light if taken in connection with the present, which, however, is itself nothing but a reconnoissance upon, what appears to me, a new field of great promise.

I have ventured to call the differential equations, whose existence will be proven in this paper, by a new name. It would be very inconvenient to characterize them in every case by the enumeration of their properties. The name Linearoid suggests at once their relation to linear differential equations.

§1.—*Existence of Linearoid Differential Equations.*

By a system of linearoid differential equations we understand a system of differential equations

$$D_i(y_1, y_2, \ldots, y_n) = 0 \qquad (i = 1, 2, \ldots, n) \quad (1)$$

with the following property: Let $y_1, y_2, \ldots, y_n$ represent a system of particular solutions; then the general solutions of (1) are obtained in the form

$$\eta_i = \sum_{k=1}^{n} \phi_{ik}(x; a_1, a_2, \ldots, a_r) y_k, \qquad (i = 1, 2, \ldots, n) \quad (2)$$

where $a_1, a_2, \ldots, a_r$ are arbitrary constants and ϕ_{ik} are uniform functions with respect to x.

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* E. J. Wilczynski, "On Systems of Multiform Functions which belong to a Group of Linear Substitutions with Uniform Coefficients." Amer. Jour. Math., April, 1899.

Our equations may then be said to have fundamental solutions. However, there is this difference between our equations and those commonly said to have fundamental solutions, that in the latter the general integrals are

$$\eta_i = \Phi_i(y_1, \ldots, y_n; a_1, \ldots, a_r),$$

where Φ_i is independent of x. The linearoid equations are very general, and in fact if we did not suppose that ϕ_{ik} are uniform functions of x, would comprehend a very large class of differential equations.

It is of course assumed that $\eta_1, \ldots, \eta_n$, the general solutions, cannot be expressed in the form (2) in terms of less than n functions $y_1, \ldots, y_n$ or less than r constants, so that r is the order of the system, and there will be no relation verified of the form

$$\sum_{k=1}^{n} \phi_{ik}(x; b_1, b_2, \ldots, b_r)\, y_k = 0, \tag{3}$$

with non-vanishing coefficients. For if there were, $\eta_1, \ldots, \eta_n$ could be expressed in terms of only $n-1$ y's.

Assuming no relation of the form (3) to be verified, $y_1, \ldots, y_n$ will be called a *fundamental system.*

Equations (2) *then define an r-parameter group G of continuous transformations.*

For give special but arbitrary values to $a_1, \ldots, a_r$. Then $\eta_1, \ldots, \eta_n$ are solutions of (1), provided $y_1, \ldots, y_n$ are such solutions. Then

$$\zeta_i = \sum_{k=1}^{n} \phi_{ik}(x; b_1, \ldots, b_r)\, \eta_k \qquad (i = 1, \ldots, n)$$

are also solutions of (1), whatever the values of the constants b_i may be. Therefore, since (2) gives the general solutions of (1), it must be possible to find $c_1, \ldots, c_r$ in terms of $a_1, \ldots, a_r$ and $b_1, \ldots, b_r$ such that

$$\zeta_i = \sum_{k=1}^{n} \phi_{ik}(x; c_1, \ldots, c_r)\, y_k. \qquad (i = 1, 2, \ldots, n)$$

But this proves the theorem.

If $y_1, \ldots, y_n$ *form a fundamental system,* $\eta_1, \ldots, \eta_n$ *will also form a fundamental system, if the determinant*

$$\Delta = |\phi_{ik}(x; a_1, a_2, \ldots, a_r)|$$

does not vanish identically.

For suppose an equation of the form

$$\sum_{k=1}^{n} \phi_{ik}(x;\ \lambda_1, \ldots, \lambda_r)\, y_k = 0$$

were verified. Then, owing to (2),

$$\sum_{k=1}^{n} \sum_{h=1}^{n} \phi_{ik}(x;\ \lambda_1, \lambda_2, \ldots, \lambda_r)\, \phi_{kh}(x;\ a_1, a_2, \ldots, a_r)\, y_h = 0\,;$$

but the coefficient of y_h can be written in the form $\phi_{ih}(x;\ c_1, \ldots, c_r)$ owing to the group property. Therefore, it must vanish, i. e.

$$\sum_{k=1}^{n} \phi_{ik}(x;\ \lambda_1, \ldots, \lambda_r)\, \phi_{kh}(x;\ a_1, \ldots, a_r) = 0, \qquad (h = 1, 2, \ldots, n)$$

whence, since $\Delta \neq 0$, it follows that

$$\phi_{ik}(x;\ \lambda_1, \ldots, \lambda_r) = 0. \qquad (k = 1, 2, \ldots, n)$$

Therefore, $\eta_1, \ldots, \eta_n$ form a fundamental system.

We have assumed the functions $\phi_{ik}(x;\ a_1, \ldots, a_r)$ to be uniform with respect to x. Let them be analytic functions of $a_1, \ldots, a_r$ likewise. Then we may assume that $\phi_{ik}(x;\ a_1, \ldots, a_r)$ can be developed in a series of powers of $a_1, \ldots, a_r$ convergent in a certain domain. The coefficient of a_λ in this development, or

$$\left(\frac{\partial \phi_{ik}}{\partial a_\lambda}\right)_0 = \psi_{ik}^{(\lambda)}(x),$$

is a uniform function of x. The infinitesimal transformations of the group G are therefore

$$U_\lambda f = \sum_{i=1}^{n} \frac{\partial f}{\partial y_i} \sum_{k=1}^{n} \psi_{ik}^{(\lambda)}(x)\, y_k, \qquad (\lambda = 1, 2, \ldots, r) \quad (4)$$

or putting

$$\xi_{\lambda i} = \sum_{k=1}^{n} \psi_{ik}^{(\lambda)}(x)\, y_k, \quad U_\lambda f = \sum_{i=1}^{n} \xi_{\lambda i} \frac{\partial f}{\partial y_i} \qquad (5)$$

Suppose for a moment that the r equations $U_\lambda f = 0$ are all independent. Denote by $G^{(s)}$ the group obtained by extending G s times. Then the r equations $U_\lambda^{(s)} f = 0$ are also independent. If $(s+1)n - r = n$, then $G^{(s)}$ will have just n independent invariants, and $sn = r$; i. e. the total order of these differential invariants of G is just equal to the number r of parameters in the group.

For every one of these differential invariants contains derivatives of order s. Otherwise there would be invariants of $G^{(s-1)}$, which is impossible, since $sn-r=0$.

Suppose now that $sn-r=l$, where $l>0$ but $<n$. Then $G^{(s-1)}$ has just $l<n$ invariants and $G^{(s)}$ has $n+l$. Each of the l invariants $\vartheta_1, \ldots, \vartheta_l$ of $G^{(s-1)}$ contains derivatives of order $s-1$, and is also an invariant of $G^{(s)}$. Their derivatives $\vartheta_1', \ldots, \vartheta_l'$ are also invariants of $G^{(s)}$, and, moreover, they are independent of each other and $\vartheta_1, \ldots, \vartheta_l$. For suppose we had for instance

$$\vartheta_1' = \frac{d\vartheta_1}{dx} = \phi\,(\vartheta_1, \vartheta_2, \ldots, \vartheta_l;\ \vartheta_2', \ldots, \vartheta_l';\ x).$$

Since ϑ_1' is a total derivative, ϕ must also be a total derivative, so that

$$\frac{d\vartheta_1}{dx} = \frac{d}{dx}\psi\,(y_1, y_2, \ldots y_n;\ \ldots;\ y_1^{(s-1)}, \ldots, y_n^{(s-1)};\ x),$$

no derivatives of order s occurring in ψ, as that would give derivatives of order $s+1$ in $\frac{d\vartheta_1}{dx}$. By integration

$$\vartheta_1 = \psi\,(y_1;\ \ldots;\ y_n;\ \ldots;\ y_1^{(s-1)} \ldots y_n^{(s-1)};\ x) + \text{const.}$$

But ϑ_1 is invariant under $G^{(s-1)}$. The same must therefore be true of ψ, whence

$$\vartheta_1 = \psi = \chi\,(\vartheta_1, \vartheta_2, \ldots, \vartheta_l;\ x).$$

But $\vartheta_1, \ldots, \vartheta_l$ being independent, χ and therefore ψ must reduce to ϑ_1, so that the supposed relation

$$\frac{d\vartheta_1}{dx} = \frac{d\psi}{dx}$$

reduces to the identity $\vartheta_1' = \vartheta_1'$.

Now let us omit the invariants $\vartheta_1', \ldots, \vartheta_l'$ and take the other n. They are independent, and none of them can be obtained from another in the set by differentiation. The total order of the system will be $(n-l)s + l(s-1) = ns - l$. But we had $sn-r=l$, so that $sn-l=r$. In this case also the total order of the system of invariants is equal to the number of parameters in the group. Moreover, any other system of invariants of $G^{(s)}$ is of higher order than r, if it contains no invariants which can be obtained from one of the others by differentiation.

Suppose now that only $q<r$ of the equations $U_\lambda f=0$ are independent. Then if $(s+1)n-r=n+l$, $G^{(s)}$ will have at least $n+l$, and $G^{(s-1)}$ at least l

invariants. Thus we can as before construct at least one system of n independent invariants of total order r. For if $G^{(s-1)}$ has $\lambda \geqq l$ independent invariants, just λ of the invariants of $G^{(s)}$ are the derivatives of these. Suppress l of the latter and replace them by the corresponding l invariants of $G^{(s-1)}$. Take any $n-l$ of the other invariants of $G^{(s)}$. The order of such a system is $l(s-1)+(n-l)s = ns-l$. But $ns-r=l$, so that

$$ns-l=r.$$

In every case then, at least one system of independent differential invariants of total order r can be found. Let $\vartheta_1, \ldots., \vartheta_n$ denote such a system, in which $\vartheta_1, \ldots., \vartheta_n$ are independent, and also of such a nature that none of these expressions can be formed from one of the others by differentiation. Putting then

$$\vartheta_1 = r_1(x), \ldots., \vartheta_n = r_n(x), \tag{6}$$

where $r_k(x)$ are functions of x alone, we have a system of differential equations which remains unaltered by every transformation of the r-parameter group G. Therefore, if $y_1, \ldots., y_n$ constitute a fundamental system, the expressions

$$\eta_i = \sum_{k=1}^{n} \phi_{ik}(x;\, a_1, \ldots., a_r)\, y_k \quad (i=1.\ 2, \ldots., n) \tag{7}$$

are also solutions of (6). Moreover, they are the general solutions, for they contain r arbitrary constants, and r is the order of the system.

This completes the proof of the existence theorem. *If the equations* (7) *form an r-parameter group, there always exists at least one system of differential equations of the* r^{th} *order, whose general solution is given by these equations,* $y_1, \ldots, y_n$ *being supposed to form a fundamental system.*

It must be noticed, however, that these differential equations are not necessarily algebraic, or still less, rational. *The linearoid differential equations have fixed branch-points,* i. e. the position of the branch-points is independent of the values of the constants of integration.

For, according to (7), since ϕ_{ik} are uniform functions of x, the general solutions $\eta_1, \ldots., \eta_n$ have the same branch-points as the special solution $y_1, \ldots., y_n$, i. e. the position of the branch-points is the same for all solutions, and therefore independent of the constants of integration.

We have

$$\xi_{\lambda k} = \psi^{(\lambda)}_{k1}(x)\,y_1 + \psi^{(\lambda)}_{k2}(x)\,y_2 + \ldots\ldots + \psi^{(\lambda)}_{kn}(x)\,y_n\,,$$

$$U_\lambda f = \sum_{k=1}^{n} \frac{\partial f}{\partial y_k}\,\xi_k \qquad (\lambda = 1,\,2,\,\ldots\ldots,\,r)$$

and

$$Vf = 0\,\frac{\partial f}{\partial x}$$

as the infinitesimal transformations of the group G, since the variable x is transformed only by the identical transformation. Such transformations generate a group if the commutators $(U_\lambda,\,U_\mu)$ can be expressed linearly in terms of $U_1,\,\ldots\ldots,\,U_r$ with constant coefficients, i. e. with coefficients independent of $y_1,\,\ldots\ldots,\,y_n$, the parameters and x. Of course

$$(U_\lambda,\,V) = 0.$$

We have further

$$(U_\lambda,\,U_\mu) = \sum_{i=1}^{n}\left[U_\lambda(\xi_{\mu i}) - U_\mu(\xi_{\lambda i})\right]\frac{\partial f}{\partial y_i}\,.$$

But

$$U_\lambda(\xi_{\mu i}) = \sum_{k=1}^{n}\left(\psi^{(\lambda)}_{k1}(x)\,y_1 + \ldots\ldots + \psi^{(\lambda)}_{kn}(x)\,y_n\right)\psi^{(\mu)}_{ik}(x),$$

$$U_\mu(\xi_{\lambda i}) = \sum_{k=1}^{n}\left(\psi^{(\mu)}_{k1}(x)\,y_1 + \ldots\ldots + \psi^{(\mu)}_{kn}(x)\,y_n\right)\psi^{(\lambda)}_{ik}(x),$$

Thus we obtain

$$(U_\lambda,\,U_\mu) = \sum_{i=1}^{n}\frac{\partial f}{\partial y_i}\sum_{k=1}^{n}\left[(\psi^{(\lambda)}_{k1}\psi^{(\mu)}_{ik} - \psi^{(\mu)}_{k1}\psi^{(\lambda)}_{ik})\,y_1 + \ldots\ldots + (\psi^{(\lambda)}_{kn}\psi^{(\mu)}_{ik} - \psi^{(\mu)}_{kn}\psi^{(\lambda)}_{ik})\,y_n\right]$$

$$= \sum_{\nu=1}^{r} c_{\lambda\mu\nu}\sum_{i=1}^{n}\frac{\partial f}{\partial y_i}\left(\psi^{(\nu)}_{i1}y_1 + \psi^{(\nu)}_{i2}y_2 + \ldots\ldots + \psi^{(\nu)}_{in}y_n\right),$$

if $c_{\lambda\mu\nu}$ denotes constant quantities.

For $f = y_i$, we obtain

$$\sum_{k=1}^{n}\left[(\psi^{(\lambda)}_{k1}\psi^{(\mu)}_{ik} - \psi^{(\mu)}_{k1}\psi^{(\lambda)}_{ik})\,y_1 + \ldots\ldots + (\psi^{(\lambda)}_{kn}\psi^{(\mu)}_{ik} - \psi^{(\mu)}_{kn}\psi^{(\lambda)}_{ik})\,y_n\right]$$

$$= \sum_{\nu=1}^{r} c_{\lambda\mu\nu}\left(\psi^{(\nu)}_{i1}y_1 + \psi^{(\nu)}_{i2}y_2 + \ldots\ldots + \psi^{(\nu)}_{in}y_n\right).$$

$$(\lambda,\,\mu = 1,\,2,\,\ldots\ldots,\,r;\; i = 1,\,2,\,\ldots\ldots,\,n)$$

But as this must be an identity, we obtain by equating coefficients of y_σ

$$\sum_{k=1}^{n} [\psi_{k\sigma}^{(\lambda)}\psi_{ik}^{(\mu)} - \psi_{k\sigma}^{(\mu)}\psi_{ik}^{(\lambda)}] = \sum_{\nu=1}^{r} c_{\lambda\mu\nu}\psi_{i\sigma}^{(\nu)}. \tag{8}$$
$$(\lambda,\ \mu = 1,\ 2,\ \ldots\ldots,\ r;\ i,\ \sigma = 1.\ 2,\ \ldots\ldots,\ n)$$

Only *if n^2r functions $\psi_{i\sigma}^{(\nu)}(x)$ and r^3 constants $c_{\lambda\mu\nu}$ can be found verifying these relations will the r infinitesimal transformations $U_\lambda f$ and Vf generate an r-parameter linearoid group.* Of course, the constants $c_{\lambda\mu\nu}$ must also verify the relations

$$\sum_{\sigma=1}^{r} (c_{gh\sigma}\, c_{\sigma j\tau} + c_{hj\sigma}\, c_{\sigma g\tau} + c_{jg\sigma}\, c_{\sigma h\tau}) = 0$$

and

$$c_{gh\tau} + c_{hg\tau} = 0.$$

Suppose we have found n^2r such functions and r^3 such constants. In order that the group which $U_\lambda f$ and Vf generate may have coefficients which are uniform functions of x, the functions $\psi_{ik}^{(\lambda)}(x)$ must be uniform. But this condition is not sufficient. The finite equations of the group are found by integrating the simultaneous system

$$\frac{d\eta_k}{dt} = \sum_{\lambda=1}^{r} c_\lambda \xi_{\lambda k} = \sum_{\lambda=1}^{r} c_\lambda\, [\psi_{k1}^{(\lambda)}(x)\,\eta_1 + \psi_{k2}^{(\lambda)}(x)\,\eta_2 + \ldots\ldots + \psi_{kn}^{(\lambda)}(x)\,\eta_n],$$

or

$$\frac{d\eta_k}{dt} = \sum_{i=1}^{n} [c_1\psi_{ki}^{(1)}(x) + c_2\psi_{ki}^{(2)}(x) + \ldots\ldots + c_r\psi_{ki}^{(r)}(x)]\,\eta_i = \sum_{i=1}^{n} A_{ki}\,\eta_i, \tag{9}$$
$$(k = 1,\ 2,\ \ldots\ldots,\ n)$$

with the condition $\eta_k = y_k$ for $t = 0$. We have put

$$A_{ki} = c_1\psi_{ki}^{(1)}(x) + c_2\psi_{ki}^{(2)}(x) + \ldots\ldots + c_r\psi_{ki}^{(r)}(x). \tag{10}$$
$$(k,\ i = 1,\ 2,\ \ldots\ldots,\ n)$$

The quantities A_{ki} are independent of t. From the well-known theory of such systems, it follows that in order that the general integrals may be uniform functions of x, *the characteristic determinant*

$$|\,A_{ki} - \delta_{ki}\omega\,| = 0, \tag{11}$$

where

$$\delta_{ki} = 0,\quad k \neq i;\quad \delta_{kk} = 1$$

must be reducible to a product of linear factors, in the field of the quantities A_{ki}, for all values of $c_1,\ c_2,\ \ldots\ldots,\ c_r$. This condition is necessary and sufficient. Moreover, *the coefficients of the finite equations of the groups are then uniform functions of the parameters* $c_1, \ldots, c_r$ *also.* Moreover, these formulæ show that *not only are the branch-points of linearoid differential equations fixed, but also the essentially singular points in Weierstrass's terminology.*

If the above condition is not fulfilled, these coefficients will be at the most n-valued functions of $x, c_1, \ldots, c_r$, provided that $\psi_{ik}^{(\lambda)}(x)$ are uniform functions.

§2.—*One-Parameter Groups, and certain r-Parameter Groups, all of whose Infinitesimal Transformations are Commutative.*

Let any infinitesimal transformation of the form

$$\left.\begin{aligned} Uf &= \sum_{k=1}^{n} \frac{\partial f}{\partial y_k} \xi_k, \\ \xi_k &= \psi_{k1}(x)\, y_1 + \psi_{k2}(x)\, y_2 + \ldots + \psi_{kn}(x)\, y_n \end{aligned}\right\} \tag{1}$$

be given, where $\psi_{ki}(x)$ are uniform functions of x. It will generate a one-parameter group linear in $y_1, \ldots, y_n$, with coefficients which are uniform functions of x, if the determinant

$$\begin{vmatrix} \psi_{11}(x) - \omega, & \psi_{12}(x) & , \ldots, & \psi_{1n}(x) \\ \psi_{21}(x) & , \psi_{22}(x) - \omega, & \ldots, & \psi_{2n}(x) \\ \ldots & \ldots & \ldots & \ldots \\ \psi_{n1}(x) & , \psi_{n2}(x) & , \ldots, & \psi_{nn}(x) - \omega \end{vmatrix} \tag{2}$$

reduces to a product of linear factors in the field of $\psi_{ik}(x)$.

If $\mathfrak{y}_1, \mathfrak{y}_2, \ldots, \mathfrak{y}_{n-1}$ be the $n-1$ independent solutions of $Uf = 0$, and $\mathfrak{y}_n$ a solution of $Uf = 1$, the variables $\mathfrak{y}_1, \ldots, \mathfrak{y}_n$ are transformed as follows:

$$\bar{x} = x, \quad \bar{\mathfrak{y}}_i = \mathfrak{y}_i \ (i = 1, 2, \ldots, n-1), \quad \bar{\mathfrak{y}}_n = \mathfrak{y}_n + t. \tag{3}$$

The equations

$$\mathfrak{y}_i(x, y_1, y_2, \ldots, y_n) = f_i(x), \quad \frac{d\mathfrak{y}_n}{dx} = f_n(x), \tag{4}$$

then form a system of differential equations such that if $y_1, \ldots\ldots, y_n$ form a fundamental system, the general solutions are given by

$$\eta_i = \sum_{k=1}^{n} \phi_{ik}(x, t)\, y_k, \qquad (i = 1, 2, \ldots\ldots, n) \qquad (5)$$

these being the finite equations of the group generated by (1).

Essentially the same method applies to an r-parameter group generated by r infinitesimal transformations of the form

$$U_\lambda f = f_\lambda(x)\, Uf. \qquad (\lambda = 1, 2, \ldots\ldots, n) \qquad (6)$$

These are all commutative, and have the same path-curves. The canonical form of the group becomes

$$\left.\begin{array}{ll} \bar{\mathfrak{y}}_i = \mathfrak{y}_i, & (i = 1, 2, \ldots\ldots, n-1) \\ \bar{\mathfrak{y}}_n = \mathfrak{y}_n + c_1 f_1(x) + c_2 f_2(x) + \ldots\ldots + c_r f_r(x), & \end{array}\right\} \qquad (7$$

$c_1, \ldots\ldots, c_r$ being arbitrary constants. Therefore $\mathfrak{y}_n$ is the general integral of a non-homogeneous linear differential equation of the r^{th} order whose fundamental integrals are $f_1(x), \ldots\ldots, f_r(x)$, By elimination, equations of the r^{th} order can be found for $y_1, y_2, \ldots\ldots, y_n$. But this elimination is not in general algebraically possible.

The functions $y_1, \ldots\ldots, y_n$, thus defined by differential equations, will, if the coefficients of these equations are uniform, have the following function-theoretic property. They are uniform except in the vicinity of certain points $a_1, \ldots\ldots, a_m$ in the plane, and when x describes a close curve around a_i, $y_1, \ldots\ldots, y_n$ will undergo a linear substitution with coefficients which are uniform functions of x. The group of linearoid substitutions thus obtained is a discreet numerable subgroup of the continuous group generated by $U_1 f, \ldots\ldots, U_r f$.

§3.—*The Group of Rotations.*

To give an example, let us consider the r-parameter group of rotations

$$\left.\begin{array}{l} \eta_1 = \cos \omega y_1 - \sin \omega y_2, \\ \eta_2 = \sin \omega y_1 + \cos \omega y_2, \\ \omega = c_1 f_1(x) + c_2 f_2(x) + \ldots\ldots + c_r f_r(x). \end{array}\right\} \qquad (1)$$

The canonical form of the group will be found by putting

$$r = \sqrt{y_1^2 + y_2^2}, \quad f = \operatorname{arc}\tan \frac{y_2}{y_1}.$$

The transformed group becomes

$$\rho = r, \quad \phi = f + c_1 f_1(x) + c_2 f_2(x) + \ldots\ldots + c_r f_r(x),$$

so that ϕ is the general integral of a non-homogeneous linear differential equation of the r^{th} order,

$$\phi^{(r)} + p_1 \phi^{(r-1)} + \ldots\ldots + p_r \phi = p,$$

of which f is a special solution, and such that $f_1(x), \ldots\ldots, f_r(x)$ form a fundamental system of the homogeneous equation

$$\phi^{(r)} + p_1 \phi^{(r-1)} + \ldots\ldots + p_r \phi = 0.$$

The uniform functions $f_1(x), \ldots\ldots, f_r(x)$ being given, the coefficients p_λ of this equation, are easily determined. They are uniform functions of x. Further, $r^2 = y_1^2 + y_2^2 = s(x)$ may be chosen as a uniform function of x. For the determination of η_1 and η_2, we have then the two equations

$$\left.\begin{aligned} &\sum_{k=0}^{r} p_k \frac{d^{r-k}}{dx^{r-k}} \operatorname{arc}\tan \frac{\eta_2}{\eta_1} = p, \qquad p_0 = 1 \\ &\eta_1^2 + \eta_2^2 = s(x), \end{aligned}\right\} \tag{3}$$

where we will, moreover, now assume that the coefficients p, p_k and $s(x)$ are rational functions of x. If y_1, y_2 form a fundamental system of (3), the general solution is given by (1).

If y_1, y_2 is a fundamental system of (3), let a be a branch-point of these functions. If x describes a circuit around a, y_1 and y_2 must undergo a linear substitution of the form (1). For all possible closed paths y_1 and y_2 will undergo a group of linear substitutions, which is a subgroup of (1).

In particular, let $f_\lambda(x) = x^\lambda$ for $\lambda = 0, 1, 2, \ldots\ldots, r-1$. The differential equations then assume the simple form

$$\left.\begin{aligned} &\frac{d^r}{dx^r} \operatorname{arc}\tan \frac{\eta_2}{\eta_1} = r(x), \\ &\eta_1^2 + \eta_2^2 = s(x) \end{aligned}\right\} \tag{4}$$

$r(x)$ and $s(x)$ being rational functions of x. Let a_i be a pole of $r(x)$ such that

$$r(x)=\sum_{i=1}^{m}\left[\frac{A_{ir}}{(x-a_i)^r}+\frac{A_{i,\,r-1}}{(x-a_i)^{r-1}}+\,\ldots\ldots+\frac{A_{i1}}{x-a_i}\right]+r'(x),$$

where $r'(x)$ contains all terms of $r(x)$ which, after r-fold integration, give a rational integral. $r'(x)$ may then still contain any number of poles, but each of these is of order higher than r. Integrating r times, we obtain

$$\begin{aligned}\text{arc tan}\,\frac{\eta_2}{\eta_1}&=\frac{1}{2\pi i}\sum_{i=1}^{m}\log(x-a_i)[\lambda_{i0}+\lambda_{i1}x+\,\ldots\ldots+\lambda_{ir-1}x^{r-1}]+\rho(x)\\&=\sigma(x),\end{aligned}\tag{5}$$

where $\rho(x)$ is a rational function of x. By properly choosing the constants A_{ik}, the constants λ_{ik} may be made to assume any arbitrarily assigned values, as may be easily verified. But if x makes a circuit around a_i, $\sigma(x)$ is increased by $\lambda_{i0}+\lambda_{i1}x+\,\ldots\ldots+\lambda_{i,\,r-1}x^{r-1}$. Therefore,

$$\eta=\frac{\eta_2}{\eta_1}=\tan\sigma(x),\tag{6}$$

is uniform everywhere except in the vicinities of the points a_i. When λ makes a positive circuit around a_i, η suffers a projective substitution of the form

$$\bar{\eta}=\frac{\eta+\tan\phi_i(x)}{1-\eta\tan\phi_i(x)},$$

where $\phi_i(x)$ are arbitrary polynomials of degree $r-1$. η_1 and η_2 undergo the corresponding homogeneous substitutions. In fact from (6) and (4) we find

$$\eta_1=\sqrt{s(x)}\cos\sigma(x),\quad \eta_2=\sqrt{s(x)}\sin\sigma(x).\tag{7}$$

If η_1 and η_2 are to have only the branch-points $a_1,\ldots\ldots,a_m$ and no others, the rational function $s(x)$ must have no zeros or poles of odd order, and the fundamental substitutions otherwise arbitrary must verify the relation

$$A_1A_2\ldots\ldots A_m=1,$$

or

$$\sum_{i=1}^{m}\lambda_{ik}=0,\quad k\neq 0,\quad \sum_{i=1}^{m}\lambda_{i0}=2\mu\pi,$$

where μ is an integer. If these relations were not verified, the point $x = \infty$ would introduce itself as a branch-point with the substitution A_{m+1} such that

$$A_1 A_2 \ldots \ldots A_m A_{m+1} = 1.$$

It is easy to verify that η_1 and η_2 are both solutions of the differential equation

$$\left[\frac{d^{r-1}}{dx^{r-1}} \frac{\frac{1}{2} s'(x) y - s(x) \frac{dy}{dx}}{2s(x) \sqrt{s(x) - y^2}}\right]^2 = r(x)^2.$$

Suppose for simplicity that there are only three branch-points 0, 1 and ∞. Then, according to a general theorem of M. Poincaré, if we put

$$x = \phi(\zeta),$$

where $\phi(\zeta)$ denotes the elliptic modular function,

$$\eta = \tan \sigma\, [\phi(\zeta)] = \psi(\zeta)$$

will be a uniform function of ζ. Moreover, if ζ undergoes a substitution of the group of the modular function, η will suffer a projective substitution whose coefficients are uniform functions of x and therefore automorphic uniform functions of ζ. Of course we obtain in the same way also a system of two uniform functions of ζ which, when ζ undergoes a substitution of the modular group, undergoes an orthogonal homogeneous substitution whose coefficients are automorphic functions of ζ.

This is, so far as I know, the first example of a new kind of homomorphic function, excepting the simple case of Poincaré's Thetafuchsian and Thetakleinian functions.

Generally there are thus brought to our attention systems of uniform functions of a variable ζ which undergo a group of homogeneous linear or projective substitutions with coefficients which are automorphic functions of ζ, when ζ suffers a group of projective substitutions with constant coefficients.

The existence of an extensive class of such functions can be established by the methods of M. Poincaré making use of the Zetafuchsian series, and numerous other examples can be found.

§4.—*Conclusion.*

In my previous paper in the American Journal, a theorem is proven which may appear to be in contradiction with some of the results of the present paper. It is there shown that functions behaving as the solutions of linearoid differential equations, verify homogeneous differential equations, under certain conditions. The equations which we have found, however, are not homogeneous. The reason for this is simply that in the former paper the domain of rationality was formed by the coefficients of the fundamental substitutions, and here a different realm is employed.

Our results serve only as an introduction to what seems to me quite a new field. An extensive class of differential equations comes before us, for which a general theory becomes possible. The explicit determination of linearoid equations and their closer discussion would appear to be a matter of great importance and interest. This can in many cases be done without any integration. It is true, as will be shown in a subsequent paper, that most of these equations can be reduced to a combination of linear differential equations and quadratures. This, of course, is in itself interesting, and shows that only the simplest cases of Λ functions are obtained by the methods of this paper, which is but an application of *Lie's* theory of finite continuous groups, which can be at once generalized for other groups. It is essential, however, for the function-theoretic application that the independent variable x is not transformed at all.

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